Trees and CFGs

Discrete Structures (CS 173) Lecture B

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Based on slides by Derek Hoiem, University of Illinois

Last class: recursive functions

$$f(n) = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$f(1) = 1$ ← base case

$f(n) = f(n-1) + n$ ← recursive formula

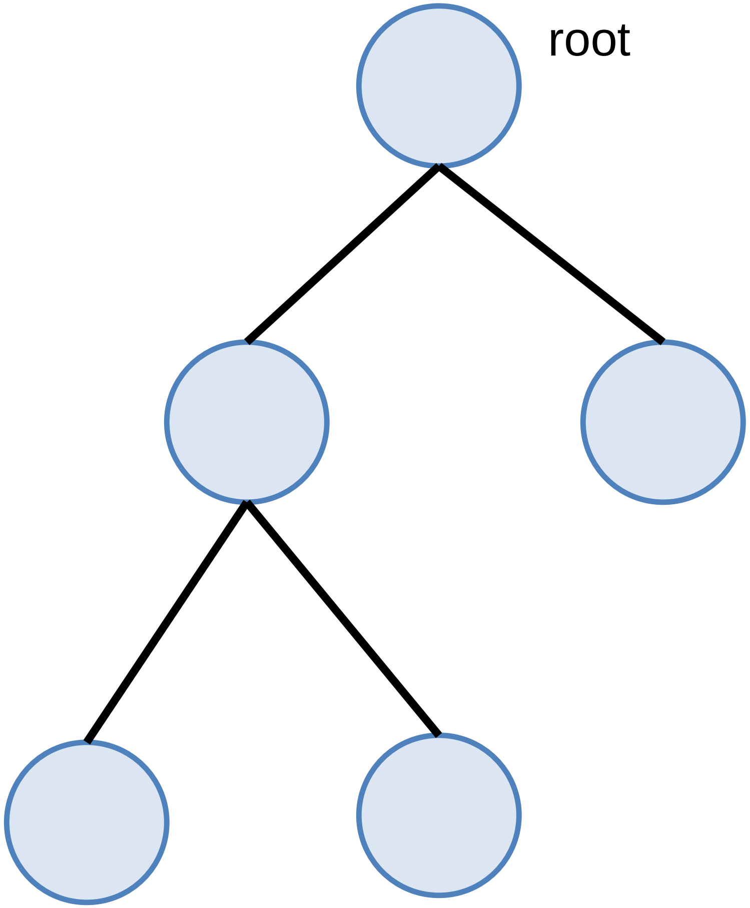
- Process for finding closed form
  1. Unroll for several steps
  2. Write in terms of $n - k$ or n/k
  3. Substitute for value of k that is base case
  4. Substitute base case value(s) and solve

- Verifying closed form with induction

Today's lecture: Trees and CFGs

- Trees
  - Examples of uses
  - Terminology
  - Induction on trees
- Context free grammars (CFGs)
  - What they are, how they work
  - Induction on CFG

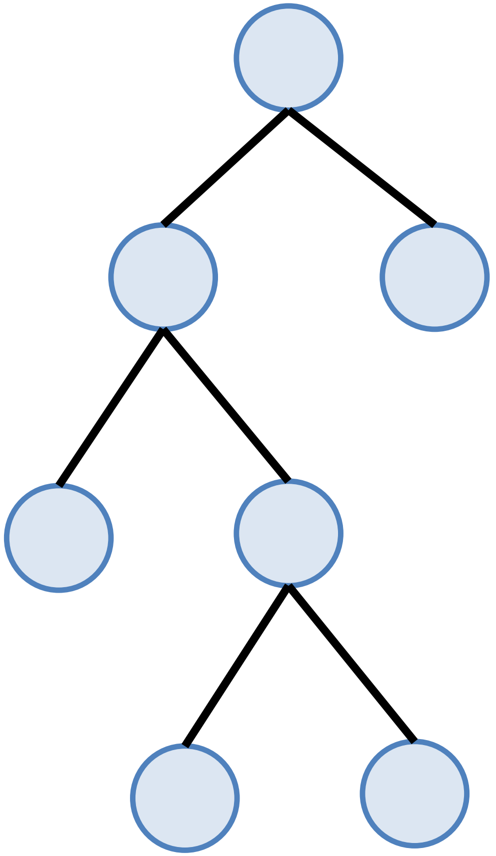
Tree: special form of graph with “root” and no cycles



Tree terminology

Nodes: root, internal, leaf, level, tree height

Relations: parent/child/sibling, ancestor/descendant



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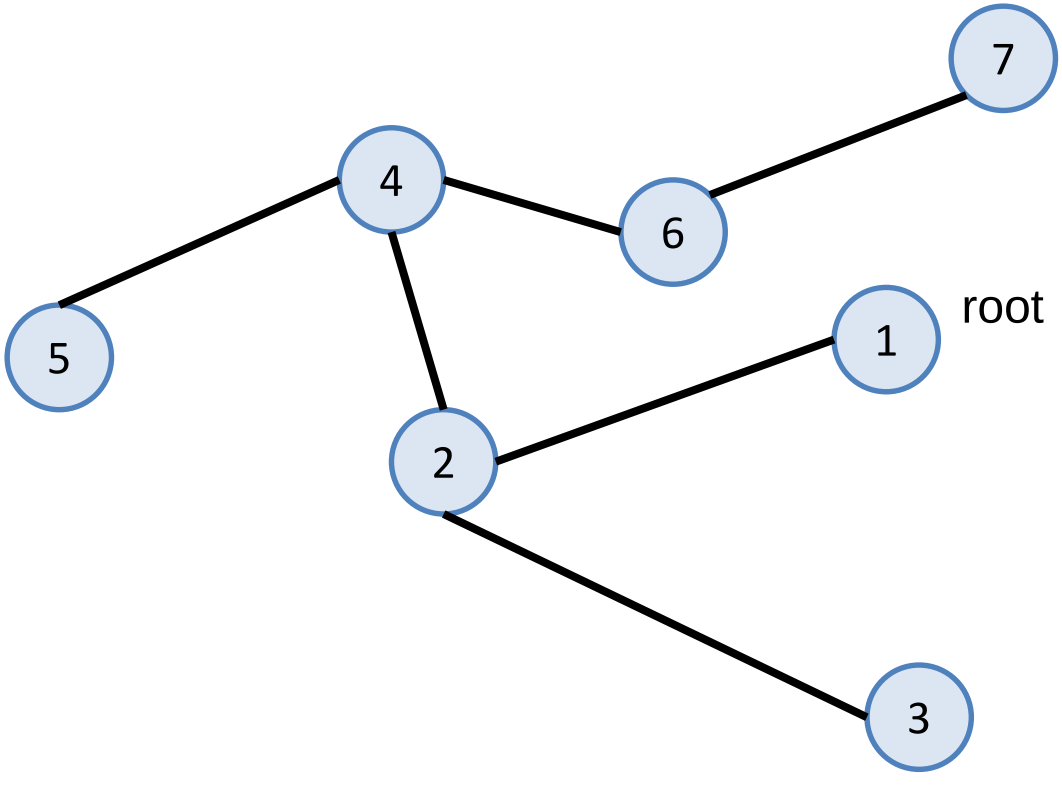
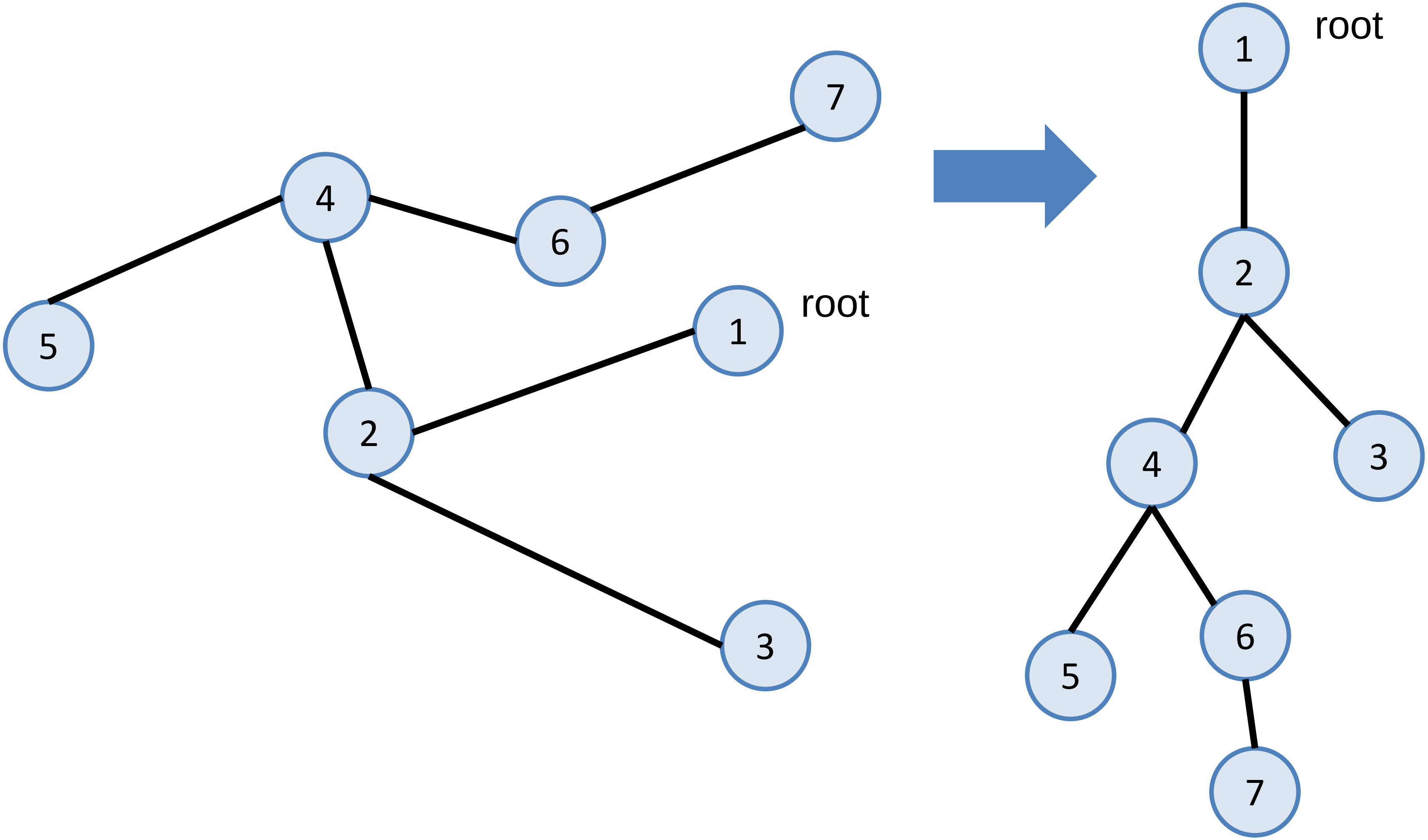
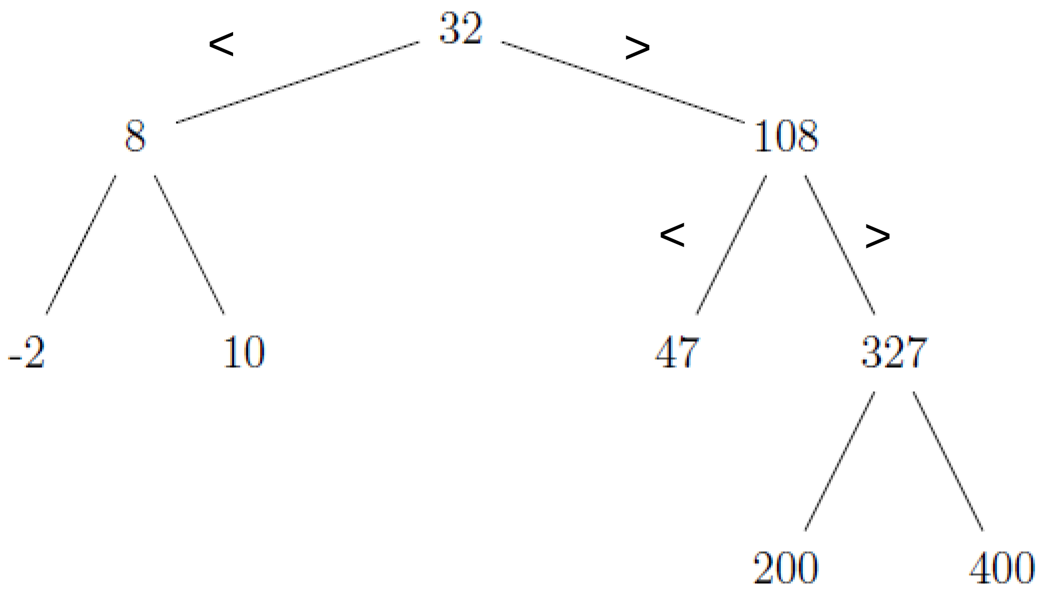
Another example



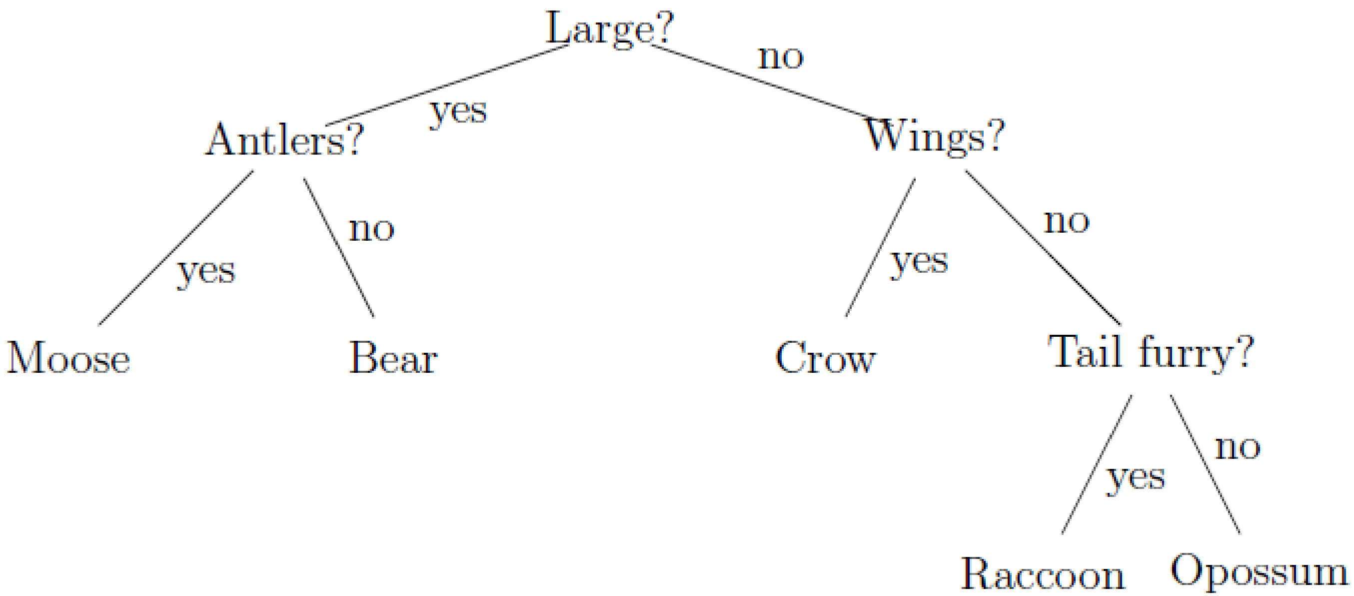
Another example



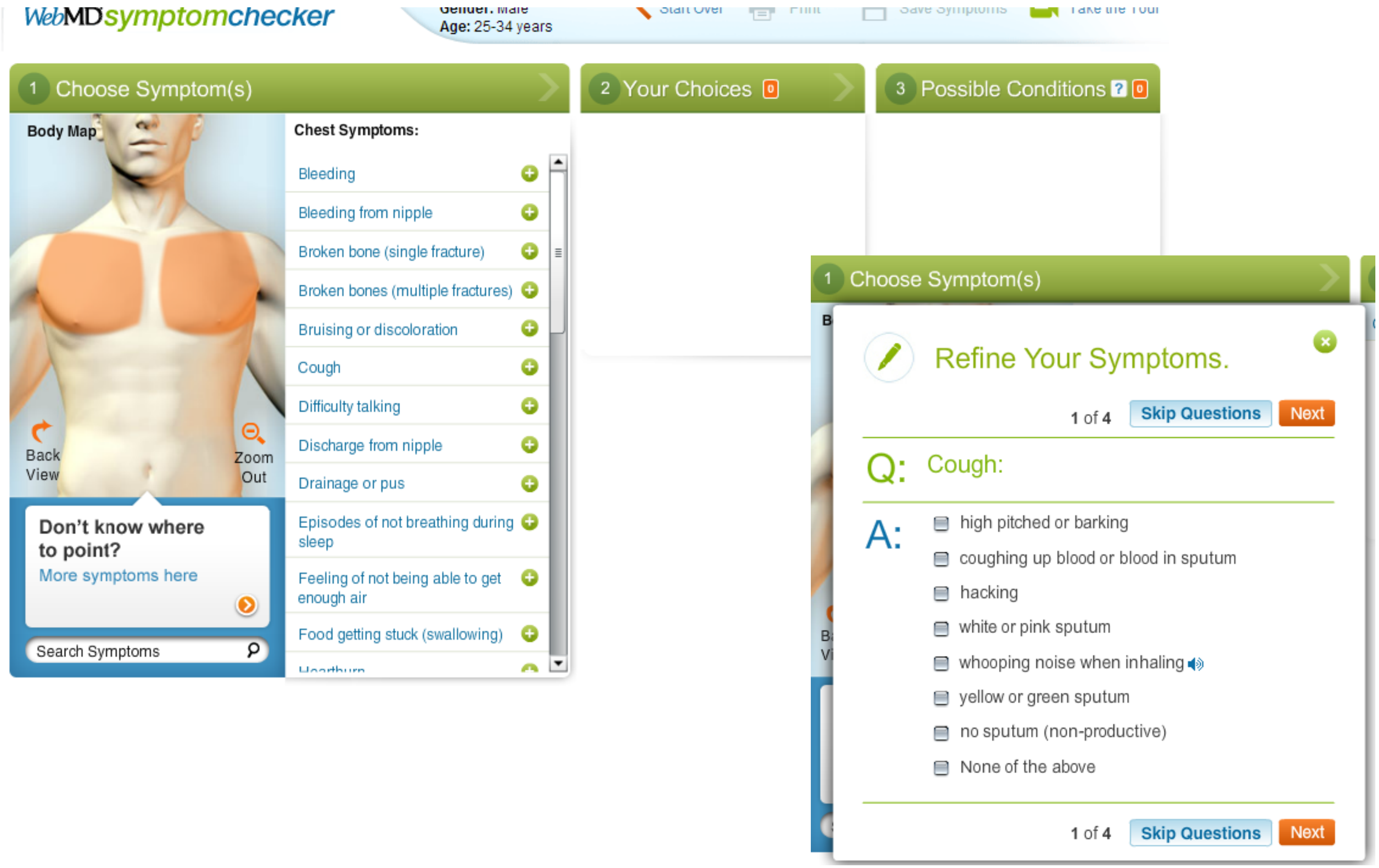
Trees for sorting



Decision trees

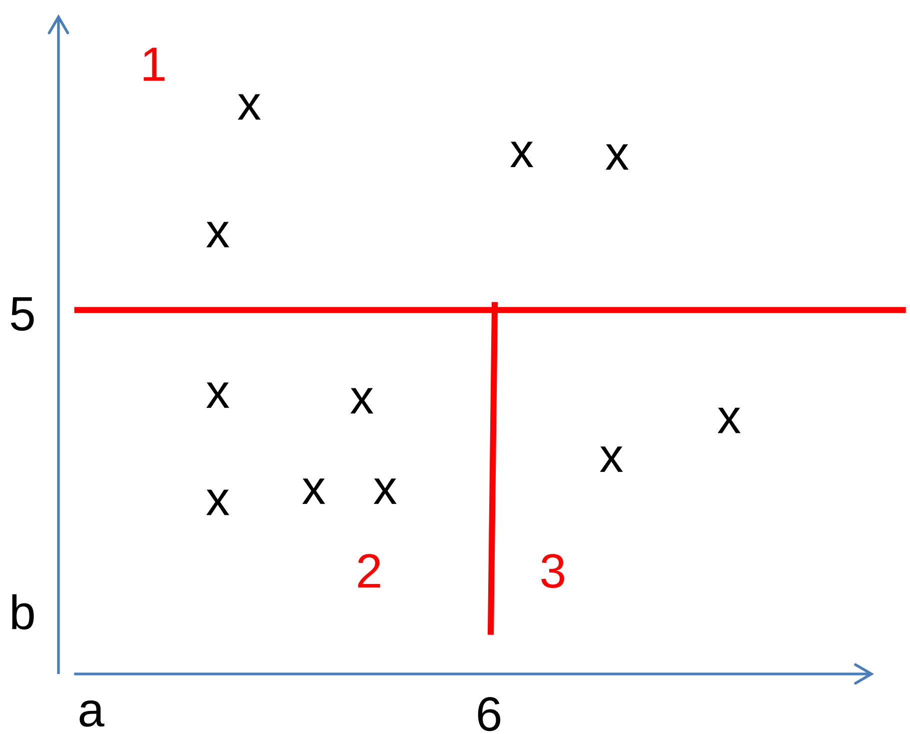


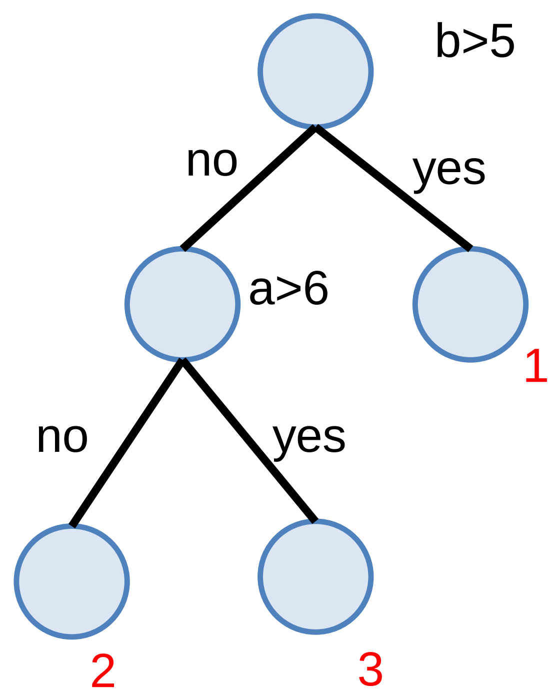
Hierarchical data structure



Trees for clustering

- Goal: create a function that maps from R^N to Z^+ such that nearby N-dimensional points are mapped to the same integer





More terminology

- m-ary tree: each node can split into m subtrees
- full: each node splits 0 or m times
- complete: all leaves have the same height
- Balanced: all leaves are “approximately” the same height
- How many nodes in a full m-ary tree with i internal nodes?
- What are the lower/upper bounds on the number of nodes n of an m-ary tree with height h?

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Induction proof on trees

Claim: In a binary tree of height h, the number of nodes $n \leq 2^{h+1} - 1$.

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Context-free Grammars

A **context-free grammar (CFG)** is a set of rules that defines a set of possible **parse trees.**

A CFG specifies a set of rules, valid start symbols, and valid terminals.

Example:

$S \rightarrow S\ a$

$S \rightarrow a \mid b \mid c$

Start symbol: S

Terminals: a, b, c

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CFG Example

Example:

$S \rightarrow b\ S\ a$

$S \rightarrow a \mid b \mid c\ a$

Which of these strings can be generated by the grammar above?

a $\quad$ bba $\quad$ ba $\quad$ $abbcaa$

bca $\quad$ $bbbcaaaa$

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Examples of parse trees

Language

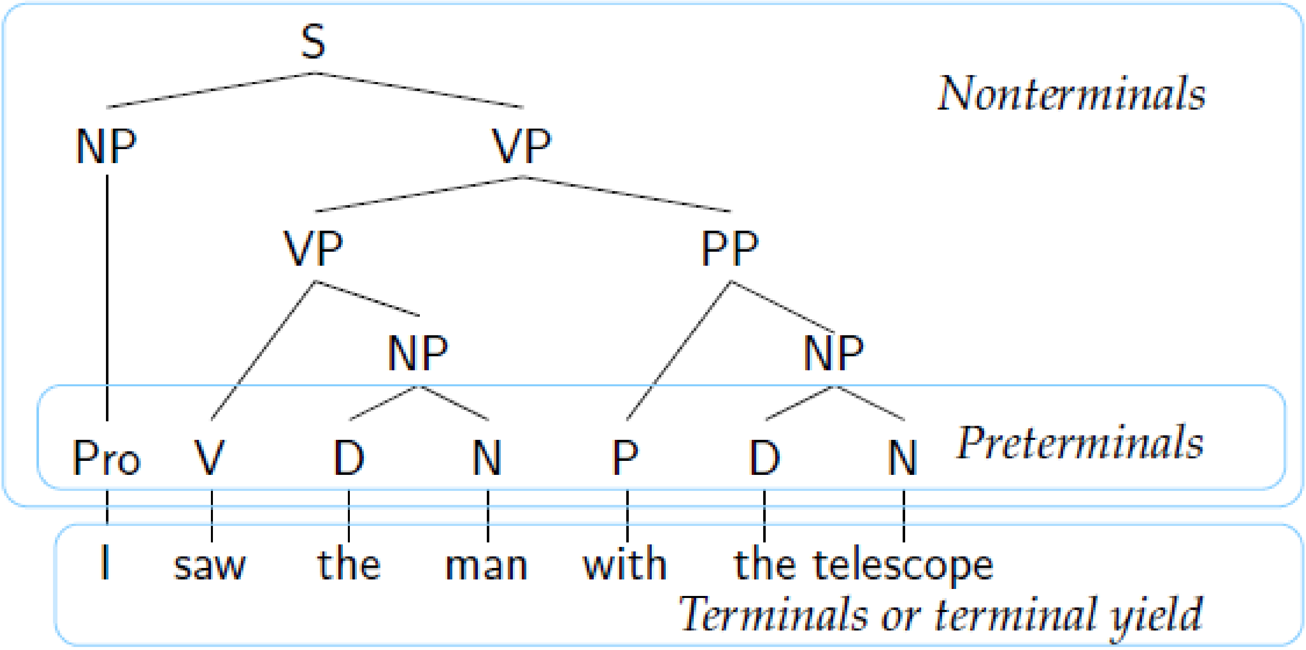
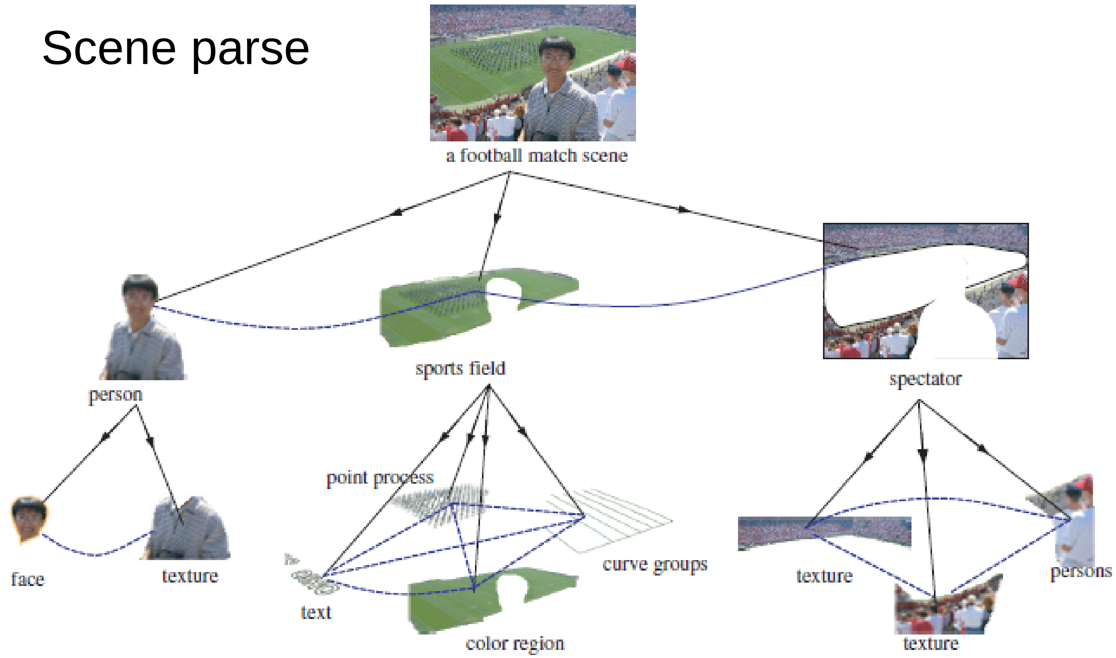


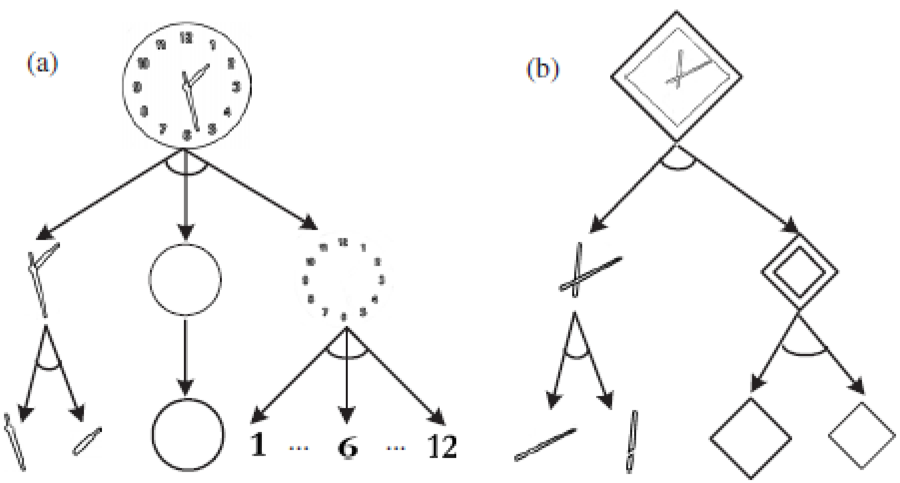
Fig: Johnson 2007

Examples of parse trees

Scene parse



Object Parse



Figs: Zhu and Mumford 2007

Examples of parse trees

Stochastic CFG for blackjack actions

Production Rules			Description
S	$\rightarrow AB$	[1.0]	Blackjack → "play game" "determine winner"
A	$\rightarrow CD$	[1.0]	play game → "setup game" "implement strategy"
B	$\rightarrow EF$	[1.0]	determine winner → "evaluate strategy" "cleanup"
C	$\rightarrow HI$	[1.0]	setup game → "place bets" "deal card pairs"
D	$\rightarrow GK$	[1.0]	implement strategy → "player strategy"
E	$\rightarrow LKM$	[0.6]	evaluate strategy → "flip dealer down-card" "dealer hits" "flip player down-card"
	$\rightarrow LM$	[0.4]	evaluate strategy → "flip dealer down-card" "flip player down-card"
F	$\rightarrow NO$	[0.5]	cleanup → "settle bet" "recover card"
	$\rightarrow ON$	[0.5]	→ "recover card" "settle bet"
G	$\rightarrow J$	[0.8]	player strategy → "Basic Strategy"
	$\rightarrow Hf$	[0.1]	→ "Splitting Pair"
	$\rightarrow bfffH$	[0.1]	→ "Doubling Down"
H	$\rightarrow l$	[0.5]	place bets
	$\rightarrow lH$	[0.5]	
I	$\rightarrow ffI$	[0.5]	deal card pairs
	$\rightarrow ee$	[0.5]	
J	$\rightarrow f$	[0.8]	Basic strategy
	$\rightarrow fJ$	[0.2]	
K	$\rightarrow e$	[0.6]	house hits
	$\rightarrow eK$	[0.4]	
L	$\rightarrow ae$	[1.0]	Dealer downcard
M	$\rightarrow dh$	[1.0]	Player downcard
N	$\rightarrow k$	[0.16]	settle bet
	$\rightarrow kN$	[0.16]	
	$\rightarrow j$	[0.16]	
	$\rightarrow jN$	[0.16]	
	$\rightarrow i$	[0.18]	
	$\rightarrow iN$	[0.18]	
O	$\rightarrow a$	[0.25]	recover card
	$\rightarrow aO$	[0.25]	
	$\rightarrow b$	[0.25]	
	$\rightarrow bO$	[0.25]	

Symbol	Domain-Specific Events (Terminals)
a	dealer removed card from house
b	dealer removed card from player
c	player removed card from house
d	player removed card from player
e	dealer added card to house
f	dealer dealt card to player
g	player added card to house
h	player added card to player
i	dealer removed chip
j	player removed chip
k	dealer pays player chip
l	player bets chip

CFG example

*$S \rightarrow NP\ \ VP$

$VP \rightarrow V\ NP \mid V\ VP \mid to\ V \mid VP\ NP \mid \epsilon$

$NP \rightarrow NP\ and\ NP \mid NP\ NP \mid N \mid \epsilon$

$V \rightarrow like \mid eat \mid drink \mid hate$

$N \rightarrow I \mid apples \mid bananas \mid coconuts \mid you$

I like to eat apples and bananas

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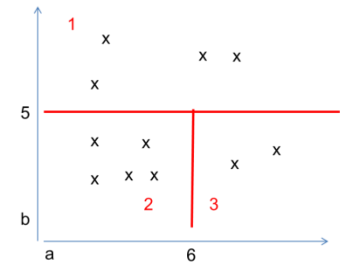
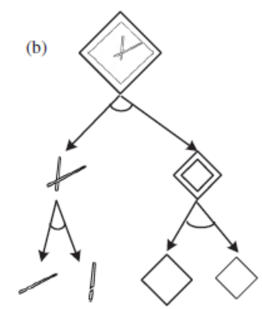
Things to remember

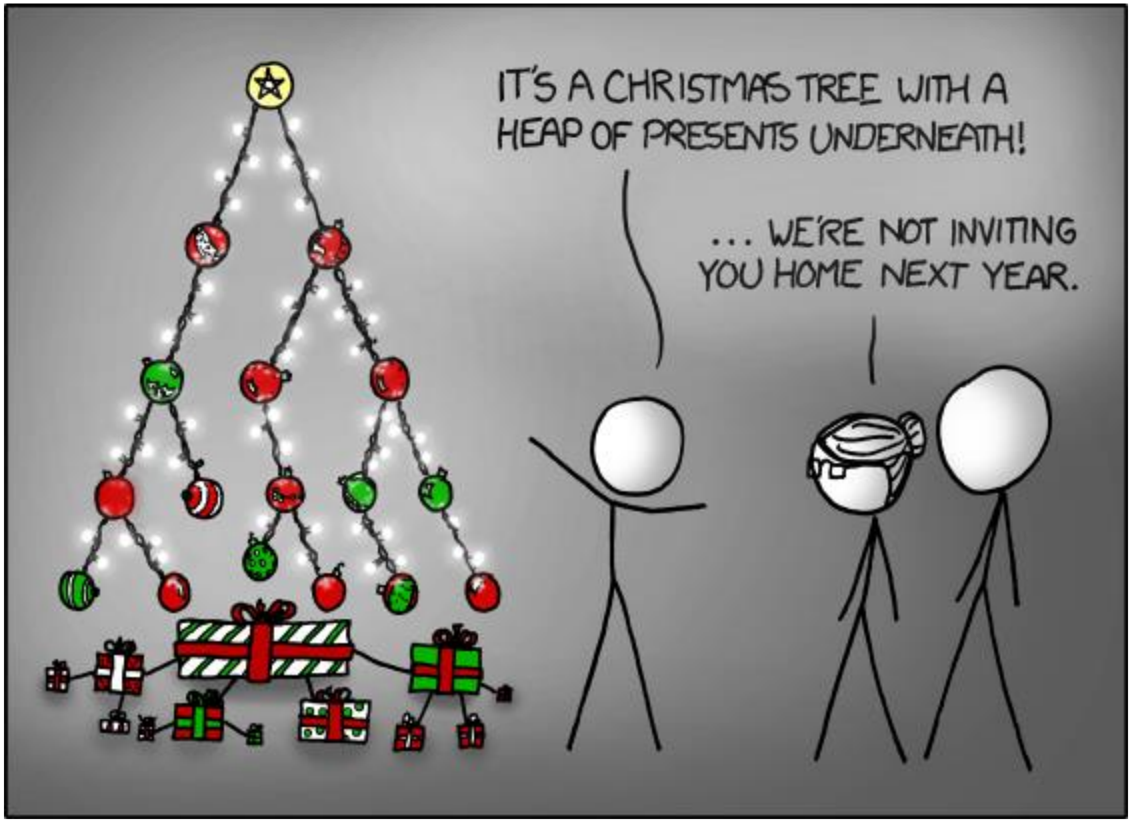
- Trees are a special graph with root and no cycles, with many uses
  - Sorting, clustering, finding similar values
  - Decision tree: machine learning, modeling choices
  - Parse trees: representing hierarchical structures





- Context free grammars: generate parse trees

- Proofs on trees: split at root, use inductive hypothesis on subtrees headed by the root's children

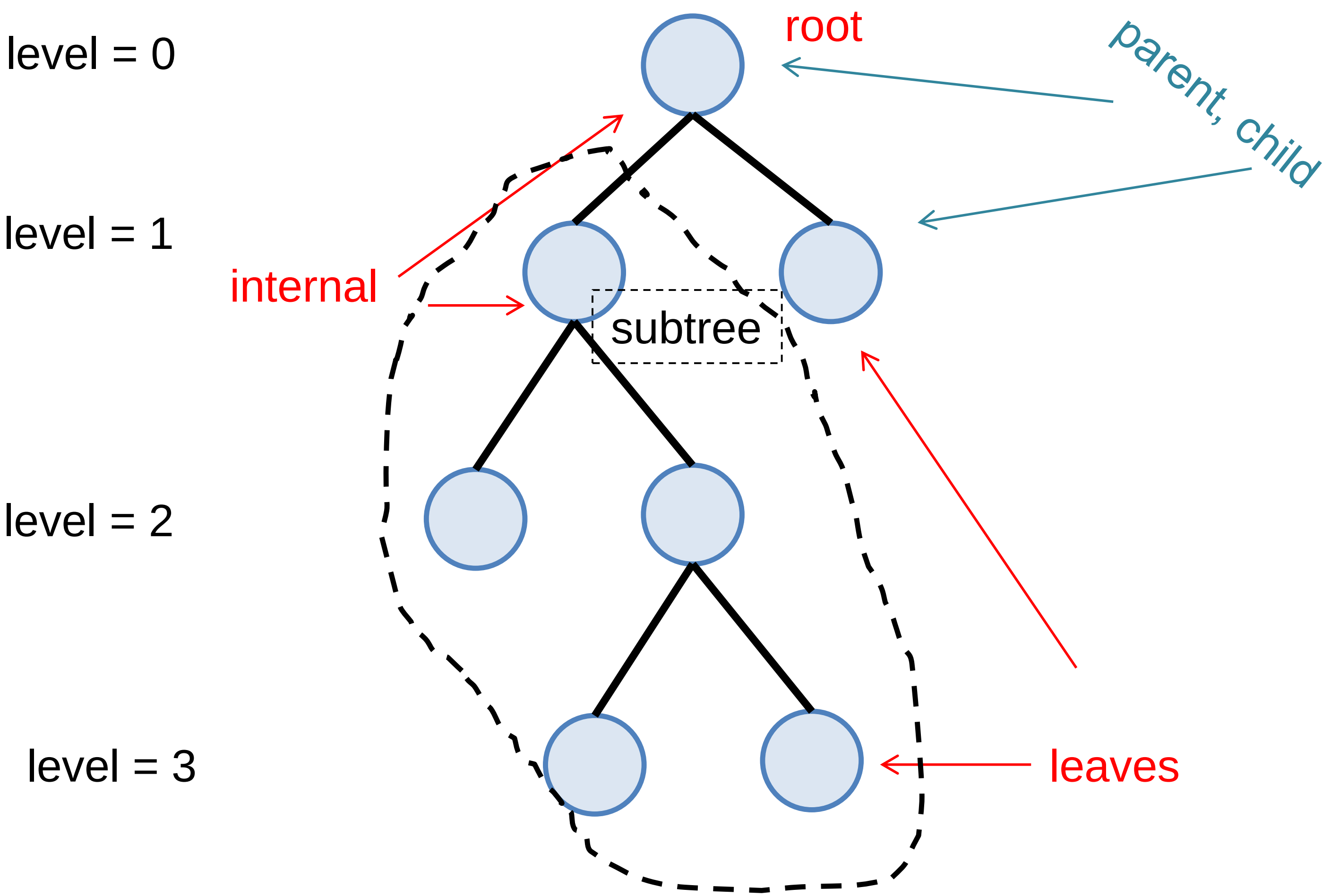

IT'S A CHRISTMAS TREE WITH A HEAP OF PRESENTS UNDERNEATH!
... WE'RE NOT INVITING YOU HOME NEXT YEAR.

xkcd

Tree terminology

Nodes: root, internal, leaf, level, tree height

Relations: parent/child/sibling, ancestor/descendant



Induction proof on CFG

$S \rightarrow a\ S\ b \mid a\ S\ S\ c$

$S \rightarrow a\ b \mid a\ c$

Claim: For any string generated by the grammar G above, the number of a's will be equal to the number of b's plus the number of c's ($n_a = n_b + n_c$)

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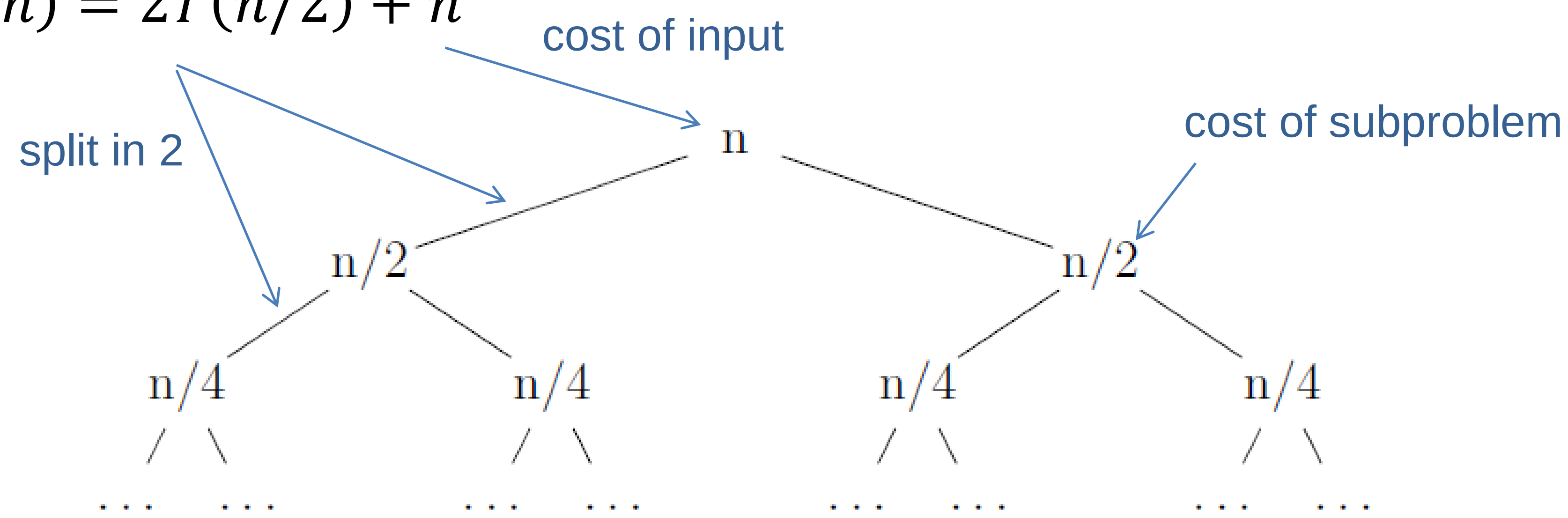
Recursion trees

$T(1) = c$

$T(n) = 2T(n/2) + n$



Cost of each level of internal nodes?

Height of tree?

Total cost of internal nodes?

Cost of leaf nodes?

Total cost = leaf cost + internal cost:

Useful formulas

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1} \qquad \sum_{k=m}^{n} r^k = \frac{r^{n+1} - r^m}{r - 1} \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} 2^k = \qquad \sum_{k=1}^{n} 2^k =$$

$$\sum_{k=0}^{n} 2^{-k} = \qquad \sum_{k=0}^{n} 2^{k+2} =$$

$$(m^a)^b = m^{ab} \qquad m^{a+b} = m^a m^b \qquad 2^{\log_2 n} = n$$

$$\log_a(b) = \log_2(b)/\log_2(a) \qquad \log(mn) = \log(m) + \log(n)$$

$$2^{\log_4(n)+2} =$$

Recursion trees

$T(1) = c$

$T(n) = 2T(n-1) + d$

Recursion trees

$\mathrm{A}(1) = c$

$\mathrm{A}(n) = 4A(n/2) + n$

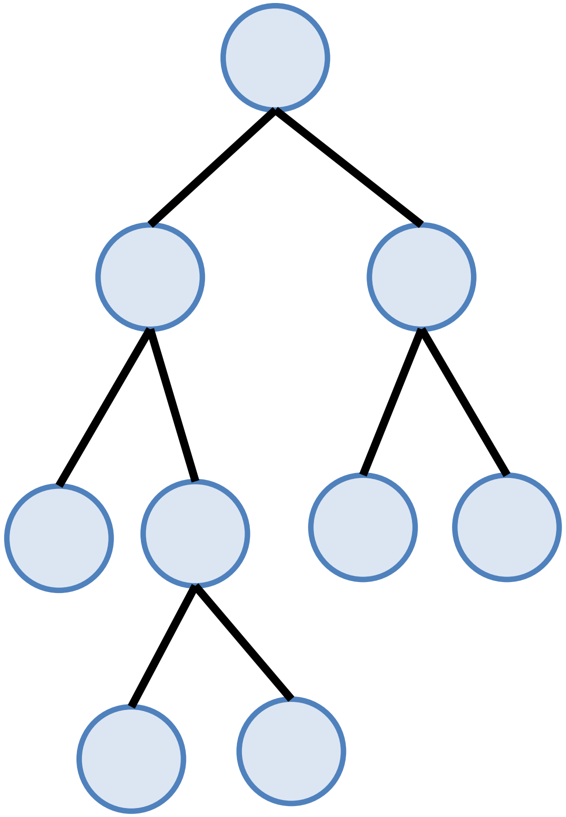
Tree induction proof

If T is a binary tree with root r, then its **rank** is

(a) 0 if r has no children

(b) $1 + q$ if r has two children, both with rank q

(c) otherwise, the maximum rank of any of the children



Tree induction proof

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(c) otherwise, the maximum rank of any of the children

Claim: A tree with rank q has at least 2^q leaves.